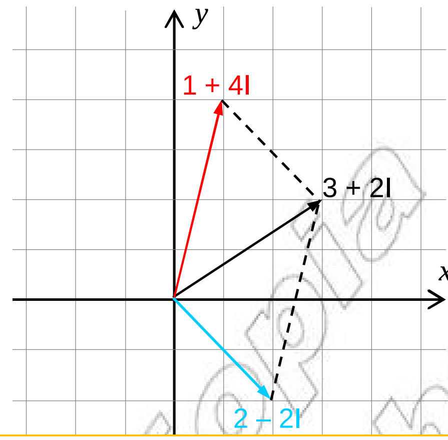
Unit 7



THE SET OF COMPLEX NUMBERS

Unit Outcomes:

After completing this unit, you should be able to:

- *know basic concepts about complex numbers.*
- *know general principles of performing operations on complex numbers.*
- *understand facts and procedures in simplifying complex numbers.*
- *show the geometric representation of complex numbers on the Argand plane.*

Main Contents

7.1 **THE CONCEPT OF COMPLEX NUMBERS**

7.2 **OPERATIONS ON COMPLEX NUMBERS**

7.3 **COMPLEX CONJUGATE AND MODULUS**

7.4 **SIMPLIFICATION OF COMPLEX NUMBERS**

7.5 **ARGAND DIAGRAM AND POLAR REPRESENTATION OF COMPLEX NUMBERS**

Key terms

Summary

Review Exercises

INTRODUCTION

Why do we need to study complex numbers?

Why do we need new numbers?

BEFORE INTRODUCING COMPLEX NUMBERS, LET US LOOK AT SIMPLE EXAMPLES THAT ILLUST
WE NEED NEW TYPES OF NUMBERS.

FOR MOST PEOPLE, "NUMBER" INITIALLY MEANT THE WHOLE NUMBERS, 0, 1, 2, 3, ... WHO
NUMBERS GIVE US A WAY TO ANSWER QUESTIONS OF THE FORM "HOW MANY…?" BUT WH
NUMBERS CAN ANSWER ONLY SOME SUCH QUESTIONS. FOR EXAMPLE, AS YOU LEARNED TO
SUBTRACT, YOU PROBABLY FOUND SOME SUBTRACTION PROBLEMS, SUCH AS $3-5$, THAT YOU COULDN'T
ANSWER WITH WHOLE NUMBERS. FURTHERMORE, YOU PROBABLY ENCOUNTERED REAL-LIFE
SUCH AS ISSUES OF TEMPERATURE AND TEMPERATURE SCALES THAT DEFIED WHOLE-NUMB
THEY SHOWED YOU THAT SUCH PROBLEMS EXIST IN REAL LIFE AS WELL AS IN THE CLASSR
THAT THEY NEED REAL ANSWERS.

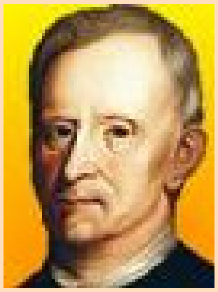
THEN YOU FOUND THAT IF YOU COULD WORK WITH INTEGERS, …, $-3, -2, -1, 0, 1, 2, 3$, …, ALL
SUBTRACTION PROBLEMS HAD ANSWERS! CLEARLY NEGATIVE NUMBERS ARE NEEDED IN REA

SO, BY USING INTEGERS, YOU CAN ANSWER ALL SUBTRACTION PROBLEMS. BUT WHAT IF
DEALING WITH DIVISION? SOME – IN FACT, *most* – DIVISION PROBLEMS DON'T HAVE INTEGER
ANSWERS. FOR EXAMPLE, $1\div2$, $3\div2$, $5\div3$ AND THE LIKE CAN'T BE ANSWERED WITH INTEGERS. SO
WE NEED NEW NUMBERS! WE THEN MOVED TO RATIONAL NUMBERS TO PROVIDE ANSWERS T
PROBLEMS.

THERE IS MORE TO THIS STORY. FOR EXAMPLE, SOME PROBLEMS REQUIRE THE USE OF SQUAR
AND OTHER OPERATIONS – BUT WE WON'T GO INTO THAT HERE. THE POINT IS THAT YO
EXPANDED YOUR IDEA OF "NUMBER" ON SEVERAL OCCASIONS, AND NOW YOU ARE ABOUT T
AGAIN.

HISTORICAL NOTE

Jean-Robert Argand



Argand was born in July 1768. He was a bookkeeper and amateur mathematician, and is remembered for having introduced the geometrical interpretation of the complex numbers as points in the Cartesian plane. His back ground and education are mostly unknown. Since his knowledge of mathematics was self thought and he did not belong to any mathematical organization, he likely pursued mathematics as a hobby rather than a profession.

OPENING PROBLEM

THE “PROBLEM” THAT LEADS TO COMPLEX NUMBERS CONCERNS SOLUTIONS OF EQUATIONS S

I $x^2 - 1 = 0$ **II** $x^2 + 1 = 0$

1 WHICH EQUATION HAS REAL ROOTS? (EQUATION I OR II?) CAN YOU EXPLAIN?

2 DRAW THE GRAPH OF $y = x^2 - 1$ AND $y = x^2 + 1$ USING THE SAME COORDINATE AXES AND IDENTIFY THE x-INTERCEPTS AND y-INTERCEPTS OF EACH GRAPH.

IN EQUATION I, -1 AND 1 ARE THE TWO REAL ROOTS. EQUATION II HAS NO REAL ROOT, SINCE THERE IS NO REAL NUMBER WHOSE SQUARE IS NEGATIVE.

DO YOU AGREE WITH THESE ANSWERS?

3 DO YOU SEE ANY DIFFERENCE BETWEEN THE x-INTERCEPTS? IF EQUATION II IS TO BE GIVEN SOLUTIONS, THEN, YOU MUST CREATE A SQUARE ROOT OF -1.

7.1 THE CONCEPT OF COMPLEX NUMBERS

IN THE ABOVE PROBLEM, FOR EQUATION II TO HAVE SOLUTIONS, YOU MUST CREATE A SQUARE ROOT OF -1.

IN GENERAL FOR ANY QUADRATIC EQUATION OF THE FORM $ax^2 + bx + c = 0$ TO HAVE SOLUTIONS, YOU NEED A NUMBER SYSTEM IN WHICH $\sqrt{b^2 - 4ac}$ IS DEFINED FOR ALL NUMBERS a, b AND c.

THE NUMBER SYSTEM WHICH YOU ARE GOING TO DEFINE IS CALLED THE **complex number system**.

TO THIS END A NEW NUMBER WHICH IS CALLED AN **“imaginary number”** NAMELY $\sqrt{-1} = i$ (READ AS **iota**) IS INTRODUCED.

Example 1 USING THE NOTATION INTRODUCED ABOVE, YOU HAVE:

A $\sqrt{-4} = \sqrt{(-1)}\sqrt{4} = 2i$ **B** $\sqrt{-25} = \sqrt{(-1)}\sqrt{25} = 5i$

C $\sqrt{-2} = \sqrt{(-1)\times 2} = \sqrt{-1}\sqrt{2} = \sqrt{2}i$

NOW YOU ARE READY TO DEFINE COMPLEX NUMBERS AS FOLLOWS:

Definition 7.1

A COMPLEX NUMBER z IS AN EXPRESSION WHICH IS WRITTEN IN THE FORM $z = x + yi$, FOR SOME REAL NUMBERS x AND y, WHERE $i = \sqrt{-1}$; THE NUMBER x IS CALLED **the real part of z** AND IS DENOTED BY $\mathrm{Re}(z)$ AND THE NUMBER y IS CALLED THE **imaginary part of z** AND IS DENOTED BY $\mathrm{Im}(z)$.

Notation:
THE SET OF COMPLEX NUMBERS DENOTED BY $\mathbb{C}$ IS GIVEN BY

$\mathbb{C} = \{z \mid z = x + yi$ WHERE x AND y ARE REAL NUMBERS; $i = \sqrt{-1}\}$

NOTE THAT $i = \sqrt{-1} \Rightarrow i^2 = -1$.

Example 2

A FOR $z = 2 - 5i$, RE(z) = 2 AND IM(z) = −5

B FOR $z = 6 + 4i$, RE(z) = 6 AND IM(z) = 4

C FOR $z = 0 + 2i = 2i$, RE(z) = 0 AND IM(z) = 2

D FOR $z = 0 + 0i = 0$, RE(z) = 0 AND IM(z) = 0

E FOR $z = 4 + 0i = 4$, RE(z) = 4 AND IM(z) = 0

Equality of complex numbers

SUPPOSE $z_1 = x + yi$ AND $z_2 = a + bi$ ARE TWO COMPLEX NUMBERS; THEN WE DEFINE THE EQUALITY OF z_1 AND z_2, WRITTEN AS $z_1 = z_2$, IF AND ONLY IF $x = a$ AND $y = b$.

Example 3 IF $15 - 3yi = 3x + 12i$, THEN $3x = 15$ AND $-3y = 12$.
THUS, $x = 5$ AND $y = -4$.

Exercise 7.1

1 WRITE THE FOLLOWING WITHOUT EXPONENTS.

A i^3 **B** i^4 **C** i^7 **D** i^8

E i^{100} **F** i^{101} **G** i^{102} **H** i^{103}

2 GENERALIZE FOR i^{2n} AND i^{2n+1}.

Hint:- *Consider the case when n is odd and when n is even.*

3 IDENTIFY THE REAL AND IMAGINARY PARTS OF EACH OF THE FOLLOWING COMPLEX NUMBERS.

A $\frac{3-5i}{7}$ **B** $\sqrt{5} + 2i\sqrt{2}$ **C** 7 **D** 5I

4 FIND THE VALUE OF THE UNKNOWNS IN EACH OF THE FOLLOWING EQUATIONS.

A $x - 3i = 2 + 12yi$ **B** $7 + 2yi = t - 10i$

5 WRITE EACH OF THE FOLLOWING REAL NUMBERS IN THE FORM $a + bi$ WHERE a AND b ARE REAL NUMBERS.

A 3 **B** −7 **C** 0 **D** $\sqrt{13}$

6 GIVEN ANY REAL NUMBER r, IS IT ALWAYS POSSIBLE TO EXPRESS r AS $a + bi$ FOR SOME REAL NUMBERS a AND b?

7 CAN YOU CONCLUDE THAT ANY REAL NUMBER IS A COMPLEX NUMBER? EXPLAIN.

7.2 OPERATIONS ON COMPLEX NUMBERS

FROM THE ABOVE EXERCISE AND THE DISCUSSIONS SO FAR, YOU CAN WRITE EVERY REAL NU r IN THE FORM OF $r = r + 0i$; THIS MEANS THAT THE SET OF REAL NUMBERS IS A SUBSET OF THE SET COMPLEX NUMBERS. NOW THE PRESENT TOPIC IS ABOUT EXTENDING THE OPERATIONS (AD SUBTRACTION, MULTIPLICATION AND DIVISION) ON THE SET OF REAL NUMBERS TO THE SET C NUMBERS.

7.2.1 Addition and Subtraction

BEFORE DEFINING ADDITION AND SUBTRACTION ON THE SET OF COMPLEX NUMBERS, LET U YOUR EXPERIENCES OF ADDING AND SUBTRACTING TERMS INVOLVING VARIABLES AS AN ACTIVITY.

ACTIVITY 7.1

PERFORM EACH OF THE FOLLOWING OPERATIONS.

A $(2x + 3y) + (5x - 7y)$ **B** $(3x + 4y) - (6x - 2y)$

C $(3 + k) + (5 - 3k)$ **D** $(5 + 4h) - (13 + 2h)$

NOW, YOU HAVE EXPERIENCE IN ADDING EXPRESSIONS SUCH AS $(3 - 5x) + (6 + 7x)$. YOU DO IT BY COMBINING SIMILAR TERMS IN THE EXPRESSIONS. FOR EXAMPLE, IF YOU WERE TO SIMPLI EXPRESSION $(3 - 5x) + (6 + 7x)$ BY COMBINING LIKE TERMS, THEN THE CONSTANTS 3 AND 6 WOULD BE COMBINED TO YIELD 9, AND THE TERMS $-5x$ AND $7x$ WOULD BE COMBINED TO YIELD $2x$. HENCE THE SIMPLIFIED FORM IS $(9 + 2x)$.

$$\text{I.E., } (3-5x)+(6+7x)=(3+6)+(-5x+7x)=9+2x$$

IN A SIMILAR FASHION, YOU COMBINE LIKE TERMS (THE REAL PART TO THE REAL PART IMAGINARY PART TO THE IMAGINARY PART) IN COMPLEX NUMBERS WHEN YOU ADD OR S FOR INSTANCE, GIVEN TWO COMPLEX NUMBERS $z_1 = 3 + 4i$ AND $z_2 = 5 + 2i$ TO FIND $z_1 + z_2$ YOU ADD 3 AND 5 TOGETHER (THE REAL PARTS) AND ADD 4 AND 2 (THE IMAGINARY PARTS) TO G AND TO FIND $z_1 - z_2$: YOU SUBTRACT 5 FROM 3 (THE REAL PARTS) AND 2 FROM 4 (THE IMAGINA PARTS) TO GET $-2 + 2i$.

Definition 7.2

GIVEN TWO COMPLEX NUMBERS $z_1 = x + yi$ AND $z_2 = a + bi$, WE DEFINE THE SUM AND DIFFERENCE OF COMPLEX NUMBERS AS FOLLOWS:

I $z_1 + z_2 = (x+a) + (y+b)i$

II $z_1 - z_2 = (x-a) + (y-b)i$

Example 1

A $(3-5i)+(6+7i)=(3+6)+(-5+7)i=9+2i$

B $(3-4i)-(2+i)=(3-2)-(4+1)i=1-5i$

Group Work 7.1

1 GIVEN $z_1 = a + bi$, $z_2 = c + di$ AND $z_3 = x + yi$, ANSWER EACH OF THE FOLLOWING:

A IS $z_1 + z_2$ A COMPLEX NUMBER? EXPLAIN. WHAT DO YOU CALL THIS PROPERTY?

B FIND $z_1 + z_2$ AND $z_2 + z_1$. IS $z_1 + z_2 = z_2 + z_1$? WHAT DO YOU CALL THIS PROPERTY?

C FIND $z_1 + (z_2 + z_3)$ AND $(z_1 + z_2) + z_3$. IS $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$? WHAT DO YOU CALL THIS PROPERTY?

D FIND $z_1 + 0, 0 + z_1, (0 = 0 + 0i)$ AND COMPARE THE VALUES.

CAN YOU CONCLUDE THAT 0 IS THE ADDITIVE IDENTITY ELEMENT?

E FIND THE SUMS $z + -z$ AND $-z + z$.

CAN YOU CONCLUDE THAT $-z$ IS THE ADDITIVE INVERSE OF z? WHY?

FROM THE ABOVE GROUP WORK, YOU CAN SUMMARIZE THE FOLLOWING:

- ✓ THE SET OF COMPLEX NUMBERS IS CLOSED UNDER ADDITION.
- ✓ ADDITION OF COMPLEX NUMBERS IS COMMUTATIVE.
- ✓ ADDITION OF COMPLEX NUMBERS IS ASSOCIATIVE.
- ✓ 0 IS THE ADDITIVE IDENTITY ELEMENT IN $\mathbb{C}$.
- ✓ FOR EVERY $z \in \mathbb{C}$ THERE IS AN ADDITIVE INVERSE $-z$ SUCH THAT $z + -z = 0 = -z + z$.

Exercise 7.2

1 PERFORM EACH OF THE FOLLOWING OPERATIONS AND WRITE YOUR ANSWER IN THE FORM OF $x + yi$.

A $\sqrt{-9}+\sqrt{-64}$ **B** $(4+5i)+(2-3i)$

C $(4+5i)-(2-3i)$ **D** $(7-11i)-(3+12i)$

E $(2+\sqrt{-16})-(1+\sqrt{-25})$ **F** i^6+i^5

G $i^{12}-i^{16}+i^{21}$ **H** $2i^9+3i^{18}$

2 SOLVE EACH OF THE FOLLOWING FOR x AND y.

A $(4-2i)+(3+5i)=x+yi$ **B** $(10+7i)-(2-3i)=x+yi$

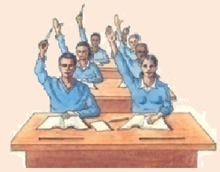
C $(x+yi)+2(3x-y)+4i=0$ **D** $(2x+3)i+4(y+4i)+5=0$

7.2.2 Multiplication and Division of Complex Numbers

Multiplication

ONCE AGAIN, BEFORE DEFINING MULTIPLICATION OF COMPLEX NUMBERS, LET US LOOK EXPERIENCE YOU HAVE IN HANDLING MULTIPLICATION CONSISTING OF TERMS WITH VARIAB ACTIVITY.

ACTIVITY 7.2



1 FIND EACH OF THE FOLLOWING PRODUCTS:

- **A** $(a+b)(a+b)$
- **B** $(a+b)(a-b)$
- **C** $(x+3y)(2x-5y)$
- **D** $(x+3)(x^2+1)$

2 USING THE FACT $i^2=-1$, FIND EACH OF THE FOLLOWING PRODUCTS:

- **A** $(2+i)(1-i)$
- **B** $(3+2i)(5+17i)$
- **C** $(3+4i)(3-4i)$
- **D** $(3+4i)(3+4i)$

Definition 7.3

GIVEN TWO COMPLEX NUMBERS $z_1 = x+yi$ AND $z_2 = a+bi$, THE PRODUCT OF z_1 AND z_2 IS DEFINED AS FOLLOWS:

$$z_1 z_2 = (ax-by)+(bx+ay)i$$

YOU DO NOT NEED TO MEMORIZE THE FORMULA, BECAUSE YOU CAN ARRIVE AT THE SAME R TREATING THE COMPLEX NUMBERS LIKE MULTIPLYING TERMS INVOLVING VARIABLES; MULTI AS USUAL AND THEN SIMPLIFY NOTING THAT $i^2=-1$.

Example 2 $(2+3i)(4+7i) = 2\times4+2\times7i+4\times3i+3i\times7i$

$$= 8+14i+12i-21 = (8-21)+(14+12)i$$

$$= -13+26i$$

Group Work 7.2

GIVEN $z_1 = a+bi$, $z_2 = c+di$ AND $z_3 = x+yi$; ANSWER THE FOLLOWING:

- **A** IS $z_1 z_2$ A COMPLEX NUMBER? EXPLAIN. WHAT DO YOU CALL THIS PROPERTY?
- **B** IS $z_1 z_2 = z_2 z_1$? WHAT DO YOU CALL THIS PROPERTY?
- **C** IS $z_1(z_2 z_3) = (z_1 z_2) z_3$? WHAT DO YOU CALL THIS PROPERTY?

D IS $z_1(z_2+z_3)=z_1z_2+z_1z_3$? WHAT DO YOU CALL THIS PROPERTY?

E IS $(z_1+z_2)z_3=z_1z_3+z_2z_3$? WHAT DO YOU CALL THIS PROPERTY?

F FIND $z_1.1$ AND $1.z_1$ $(1=1+0i)$ AND COMPARE THE VALUES.

CAN YOU CONCLUDE THAT 1 IS THE MULTIPLICATIVE IDENTITY ELEMENT?

FROM THE ABOVE ACTIVITIES YOU CAN SUMMARIZE THE FOLLOWING:

- ✓ THE SET OF COMPLEX NUMBERS IS CLOSED UNDER MULTIPLICATION.
- ✓ MULTIPLICATION OF COMPLEX NUMBERS IS COMMUTATIVE.
- ✓ MULTIPLICATION OF COMPLEX NUMBERS IS ASSOCIATIVE.
- ✓ MULTIPLICATION IS DISTRIBUTIVE OVER ADDITION IN ℂ
- ✓ 1 IS THE MULTIPLICATIVE IDENTITY ELEMENT IN ℂ

Division

YOU CAN THINK OF DIVISION AS THE INVERSE PROCESS OF MULTIPLICATION, SINCE FOR ANY T NUMBERS a AND b WITH $b \neq 0$ THE PHRASE "a IS DIVIDED BY b" CAN BE SYMBOLIZED AS:

$$\frac{a}{b}=a\left(\frac{1}{b}\right);\ b\neq 0.$$

NOW, DO THE SAME THING FOR COMPLEX NUMBERS IN THE FOLLOWING GROUP WORK.

Group Work 7.3

1 JUSTIFY EACH STEP IN THE OPERATION PERFORMED

$$\left(\frac{1}{2+3i}\right)\left(\frac{1}{2-3i}\right)=\frac{1}{13}$$

$$\frac{1}{2+3i}=\left(\frac{1}{2+3i}\right)\left(\frac{2-3i}{2-3i}\right)$$

$\frac{1}{2+3i}$ IS THE MULTIPLICATIVE INVERSE OF $2+3i$

$\frac{2}{13}-\frac{3i}{13}$ IS THE MULTIPLICATIVE INVERSE OF $2+3i$

2 GIVE REASONS FOR THE FOLLOWING ARGUMENTS.

GIVEN $z=a+bi\neq 0$ $(0=0+0i)$

$$\frac{1}{a+bi}=\left(\frac{1}{a+bi}\right)\left(\frac{a-bi}{a-bi}\right)$$

$$\frac{1}{a+bi}=\frac{a}{a^2+b^2}-\frac{bi}{a^2+b^2}$$

YOU CONCLUDE THAT $\frac{a}{a^2+b^2}-\frac{bi}{a^2+b^2}$ IS THE MULTIPLICATIVE INVERSE OF $a+bi$

NOW DIVISION OF COMPLEX NUMBERS CAN BE DEFINED AS FOLLOWS:

SUPPOSE $z_1 = x + yi$ AND $z_2 = a + bi \neq 0$ ARE GIVEN, THEN YOU HAVE THE FOLLOWING:

$$\frac{z_1}{z_2} = z_1 \frac{1}{z_2} = (x+yi)\left(\frac{1}{a+bi}\right) = (x+yi)\left(\frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2}\right)$$

$$= \frac{ax+by}{a^2+b^2} + \frac{(ay-bx)i}{a^2+b^2}$$

Definition 7.4

SUPPOSE $z_1 = x + yi$ AND $z_2 = a + bi \neq 0$ ARE GIVEN, THEN z_1 DIVIDED BY z_2 DENOTED BY $\frac{z_1}{z_2}$ OR $z_1 \div z_2$ IS DEFINED TO BE $z_1 \div z_2 = \frac{z_1}{z_2} = \frac{ax+by}{a^2+b^2} + \frac{(ay-bx)i}{a^2+b^2}$

Note:

FOR EVERY $z \neq 0$ IN ℂ THERE IS ITS MULTIPLICATIVE INVERSE $\frac{1}{z}$ SUCH THAT $z \times \frac{1}{z} = 1 = \frac{1}{z} \times z$.

Example 3

A $\frac{1}{3+7i} = \frac{3}{3^2+7^2} - \frac{7i}{3^2+7^2} = \frac{3}{58} - \frac{7i}{58}$

B $\frac{i+1}{3-4i} = (i+1)\left(\frac{3}{3^2+4^2} - \frac{(-4i)}{3^2+4^2}\right) = (i+1)\left(\frac{3}{25} + \frac{4i}{25}\right)$

$= \frac{-1}{25} + \frac{7i}{25}$

Exercise 7.3

PERFORM THE FOLLOWING OPERATIONS AND WRITE YOUR ANSWERS IN THE FORM OF $a + bi$, WHERE a AND b ARE REAL NUMBERS.

1 $(-3+4i)(2-2i)$

2 $3i(2-4i)$

3 $(2-7i)(3+4i)$

4 $(1+i)(2-3i)$

5 $(2-i) - i(1-2i)$

6 $\left(\frac{2-3i}{1-i}\right)\left(\frac{1+i}{2+3i}\right)$

7 $\frac{2-3i}{3+2i} + 6 + 9i$

8 $i^{12} - i^{7}$

9 $i^{20} - i^{24} + i^{15}$

10 $\frac{1}{2+3i}$

11 $\frac{i+3}{5-2i}$

12 $\frac{4-2i}{1-i}$

7.3 COMPLEX CONJUGATE AND MODULUS

ACTIVITY 7.3

GIVEN COMPLEX NUMBERS $z_1 = x + yi$ AND $z_2 = x - yi$ FIND

A THE PRODUCT z_1z_2 **B** THE SUM $z_1 + z_2$ **C** THE DIFFERENCE $z_1 - z_2$

FROM THE ABOVE ACTIVITY YOU CAN OBSERVE THE FOLLOWING:

I $(x + yi)(x - yi) = x^2 + y^2$ WHICH IS A REAL NUMBER.

II $(x + yi) + (x - yi) = 2x$ WHICH IS TWICE THE REAL PART.

III $(x + yi) - (x - yi) = 2yi$ WHICH IS A PURELY IMAGINARY NUMBER.

THE COMPLEX NUMBER $x - yi$ IS CALLED **the conjugate** (OR **complex conjugate**) OF THE COMPLEX NUMBER $x + yi$. CONJUGATES ARE IMPORTANT BECAUSE OF THE FACT THAT A COMPLEX NUMBER MULTIPLIED BY ITS CONJUGATE IS REAL; I.E., $(x + yi)(x - yi) = x^2 + y^2$

Definition 7.5

THE COMPLEX CONJUGATE (OR CONJUGATE) OF A COMPLEX NUMBER $z = x + yi$, DENOTED BY $\bar{z}$, IS GIVEN BY $\bar{z} = x - yi$

Example 1

A IF $z = 5 - 6i$, THEN $\bar{z} = 5 - (-6)i = 5 + 6i$

B IF $z = -1 + \frac{1}{2}i$, THEN $\bar{z} = -1 - \frac{1}{2}i$

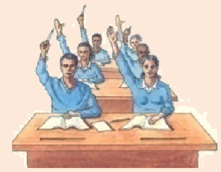
C IF $z = 4 = 4 + 0i$, THEN $\bar{z} = 4$ **D** IF $z = -2i$, THEN $\bar{z} = 2i$

Example 2 IN THE TABLE BELOW, THREE COLUMNS ARE FILLED IN; YOU ARE EXPECTED TO FILL IN THE REMAINING TWO COLUMNS.

Complex number z	Conjugate of z ($\bar{z}$)	Product ($z\bar{z}$)	Sum ($z+\bar{z}$)	Difference ($z-\bar{z}$)
$2 + 3i$	$2 - 3i$	13		
$2 - 3i$	$2 + 3i$	13		
$3 - 5i$	$3 + 5i$	34		
$3 + 5i$	$3 - 5i$	34		
$4i$	$-4i$	16		
$-4i$	$4i$	16		
5	5	25		
$a + bi$	$a - bi$	$a^2 + b^2$		
$a - bi$	$a + bi$	$a^2 + b^2$		

Properties of conjugates

ACTIVITY 7.4



GIVEN TWO COMPLEX NUMBERS $z_1 = 3 + 4i$ AND $z_2 = -5 \;\; i2$ FIND THE FOLLOWING:

A $\overline{z_1}$ **B** $\overline{z_2}$ **C** $\overline{z_1} + \overline{z_2}$

D $z_1 + z_2$ **E** $\overline{z_1 + z_2}$ **F** $\overline{z_1}\,\overline{z_2}$

G $\overline{z_1 z_2}$ **H** $\dfrac{\overline{z_1}}{\overline{z_2}}$ **I** $\overline{\left(\dfrac{z_1}{z_2}\right)}$

J $\overline{\overline{z}}_1$ **K** $\overline{\overline{z}}_2$

FROM THE ABOVE ACTIVITY YOU MAY SUMMARIZE PROPERTIES OF CONJUGATES AS FOLLOWS:

Theorem 7.1

FOR ANY COMPLEX NUMBERS z_1 AND z_2, THE FOLLOWING PROPERTIES HOLD TRUE.

I $\overline{\overline{z_1}} = z_1$ **II** $z_1 + \overline{z_1} = 2\,\mathrm{RE}(z_1)$ **III** $z_1 - \overline{z_1} = 2i\,\mathrm{IM}(z_1)$

IV $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ **V** $\overline{z_1 . z_2} = \overline{z_1} . \overline{z_2}$ **VI** $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$, IF $z_2 \neq 0$

(THE PROOF OF THIS THEOREM IS LEFT AS AN EXERCISE TO YOU.)

NOTE THAT IV AND V OF THE ABOVE THEOREM CAN BE EXTENDED TO ANY FINITE NUMBER OF TERMS. I.E.,

$$\overline{z_1 + z_2 + ... + z_n} = \overline{z_1} + \overline{z_2} + ... + \overline{z_n} \text{ AND } \overline{z_1 . z_2 ..... z_n} = \overline{z_1} . \overline{z_2} .... \overline{z_n}$$

ONE OF THE IMPORTANT USES OF A COMPLEX CONJUGATE IS TO FACILITATE DIVISION OF COMPLEX NUMBERS. AS YOU HAVE SEEN, DIVISION IS THE INVERSE PROCESS OF MULTIPLICATION.

I.E., $\dfrac{z_1}{z_2} = z_3$ IF AND ONLY IF $z_1 = z_2 . z_3$ PROVIDED $z_2 \neq 0$

IF $z_1 = x + yi$, $z_2 = a + bi$ AND $z_3 = c + di$, THEN FROM $(x + yi)\dfrac{1}{a + bi} = c + di$, ONE COULD SOLVE THE FOLLOWING:

$$x + yi = (a + bi)(c + di)$$

$$x + yi \cdot \frac{1}{a+bi} = (c+di)$$

$c = \dfrac{ax+by}{a^2+b^2}$ AND $d = \dfrac{ay-bx}{a^2+b^2}$ AND CONCLUDE THAT $z_3 = \dfrac{z_1}{z_2} = \dfrac{ax+by}{a^2+b^2} + \dfrac{ay-bx}{a^2+b^2}i$

HOWEVER, THIS IS VERY TEDIOUS! INSTEAD, YOU CAN USE CONJUGATES TO SIMPLIFY EXPRES THE FORM $(x + yi) \div (a + bi)$ BY WRITING IT IN THE FORM $\dfrac{x+yi}{a+bi}$ AND MULTIPLYING BOTH THE NUMERATOR AND DENOMINATOR BY $a - bi$, WHICH IS THE CONJUGATE OF $a + bi$ TO ARRIVE AT THE QUOTIENT.

Example 3 IF $z_1 = 2+3i$ AND $z_2 = 5-i$, THEN,

$$\frac{z_1}{z_2} = \frac{2+3i}{5-i} = \left(\frac{2+3i}{5-i}\right)\left(\frac{5+i}{5+i}\right) = \frac{7}{26} + \frac{17}{26}i$$

SO, ONE CAN CONSIDER DIVISION OF A COMPLEX NUMBER AS MULTIPLYING BOTH THE DIVIDE AND THE DIVISOR BY THE CONJUGATE OF THE DIVISOR.

Definition 7.6

THE ABSOLUTE VALUE (OR MODULUS) OF A COMPLEX NUMBER $z = x + yi$, DENOTED BY $|z|$, IS DEFINED TO BE

$$|z| = \sqrt{x^2 + y^2}$$

THIS IS A NATURAL GENERALIZATION OF THE ABSOLUTE VALUE OF REAL NUMBERS, SINCE $|x + 0i| = \sqrt{x^2} = |x|$.

Example 4

A IF $z = 2+5i$, THEN $|z| = \sqrt{2^2 + 5^2} = \sqrt{29}$

B IF $z = 5+12i$, THEN $|z| = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$

C IF $z = i$, THEN $|z| = \sqrt{1^2} = 1$

D IF $z = -2$, THEN $|z| = \sqrt{(-2)^2} = |-2| = 2$

Note:

IF $z_1 = x + yi$ AND $z_2 = a + bi$, THEN

$$|z_1 - z_2| = |(x-a) + (y-b)i| = \sqrt{(x-a)^2 + (y-b)^2}$$

SOME PROPERTIES OF CONJUGATES AND MODULUS CAN BE SUMMARIZED AS FOLLOWS:

Theorem 7.2

FOR ANY TWO COMPLEX NUMBERS z_1 AND z_2, THE FOLLOWING PROPERTIES HOLD TRUE:

I	$z_1.\bar{z}_1 = \lvert z_1\rvert^2$	V	$\lvert z_1.z_2\rvert = \lvert z_1\rvert\cdot\lvert z_2\rvert$
II	$\lvert z_1\rvert = \lvert\bar{z}_1\rvert$	VI	$\left\lvert\frac{z_1}{z_2}\right\rvert = \frac{\lvert z_1\rvert}{\lvert z_2\rvert}$, IF $z_2 \neq 0$
III	$\lvert\text{RE}(z_1)\rvert \le \lvert z_1\rvert$	VII	TRIANGLE INEQUALITY: $\lvert z_1+z_2\rvert \le \lvert z_1\rvert + \lvert z_2\rvert$
IV	$\lvert\text{IM}(z_1)\rvert \le \lvert z_1\rvert$	VIII	$\lvert z_1 - z_2\rvert \ge \lvert z_1\rvert - \lvert z_2\rvert$

Proof:

LET $z_1 = x + yi$ AND $z_2 = u + vi$ FOR SOME REAL NUMBERS x, y, u AND v.

I TO SHOW THAT $z_1.\bar{z}_1 = |z_1|^2$, SIMPLY YOU MULTIPLY z_1 WITH ITS CONJUGATE $\bar{z}_1 = x - yi$ AS FOLLOWS:

$$z_1\cdot\bar{z}_1 = (x+yi)(x-yi) = (x^2+y^2) + (x(-y)+y(x))i = x^2+y^2 = |z_1|^2$$

II TO SHOW THAT $|z_1| = |\bar{z}_1|$, SINCE $\bar{z}_1 = x - yi$, YOU HAVE

$$|\bar{z}_1| = \sqrt{x^2+(-y)^2} = \sqrt{x^2+y^2} = |z_1|$$

III TO SHOW THAT $|\text{RE}(z_1)| \le |z_1|$, SINCE $x^2 \le x^2+y^2$ FOR EVERY REAL NUMBERS x AND y, YOU HAVE

$$|\text{RE}(z_1)| = |x| = \sqrt{x^2} \le \sqrt{x^2+y^2} = |z_1|$$

IV TO SHOW THAT $|\text{IM}(z_1)| \le |z_1|$, SINCE $y^2 \le x^2+y^2$, FOR EVERY REAL NUMBERS x AND y, YOU HAVE

$$|\text{IM}(z_1)| = |y| = \sqrt{y^2} \le \sqrt{x^2+y^2} = |z_1|$$

V TO SHOW THAT $|z_1.z_2| = |z_1|\cdot|z_2|$,

$$|z_1\cdot z_2|^2 = (z_1.z_2)\cdot(\overline{z_1.z_2}) \qquad \text{BY I}$$

$$= (z_1.z_2)\cdot(\bar{z}_1.\bar{z}_2) = (z_1\cdot\bar{z}_1)\cdot(z_2\cdot\bar{z}_2)$$

$$= |z_1|^2\cdot|z_2|^2 = \left(|z_1|\cdot|z_2|\right)^2$$

$$\Leftrightarrow |z_1\cdot z_2| = |z_1|\cdot|z_2|$$

VI TO SHOW THAT $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, IF $z_2 \neq 0$,

$$\left|\frac{z_1}{z_2}\right|^2 = \left(\frac{z_1}{z_2}\right)\cdot\overline{\left(\frac{z_1}{z_2}\right)} = \frac{z_1}{z_2}\cdot\frac{\overline{z}_1}{\overline{z}_2} = \frac{z_1\cdot\overline{z}_1}{z_2\cdot\overline{z}_2} = \frac{|z_1|^2}{|z_2|^2} = \left(\frac{|z_1|}{|z_2|}\right)^2$$

$\Rightarrow \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, PROVIDED THAT $z_2 \neq 0$

VII TO SHOW THAT $|z_1 + z_2| \le |z_1| + |z_2|$,

$$|z_1 + z_2|^2 = (z_1 + z_2)\cdot\left(\overline{z_1 + z_2}\right) = (z_1 + z_2)(\overline{z}_1 + \overline{z}_2)$$
$$= z_1\cdot\overline{z}_1 + z_1\cdot\overline{z}_2 + z_2\cdot\overline{z}_1 + z_2\cdot\overline{z}_2$$
$$= |z_1|^2 + z_1\cdot\overline{z}_2 + \overline{z_1.\overline{z}_2} + |z_2|^2$$
$$= |z_1|^2 + 2\,\mathrm{RE}(z_1\cdot\overline{z}_2) + |z_2|^2,$$

BUT $2\,\mathrm{RE}(z_1\,\overline{z}_2) = 2|z_1\,\overline{z}_2| = 2|z_1||\overline{z}_2| = 2|z_1||z_2|$.

THUS $|z_1|^2 + 2\,\mathrm{RE}(z_1 z_2) + |z_2|^2 \le |z_1|^2 + 2\left|z_1\overline{z_2}\right| + |z_2|^2 = (|z_1| + |z_2|)^2$

$\Rightarrow |z_1 + z_2|^2 \le (|z_1| + |z_2|)^2$

$\Rightarrow |z_1 + z_2| \le |z_1| + |z_2|$, WHICH IS THE REQUIRED RESULT.

VIII TO SHOW THAT $|z_1 - z_2| \ge |z_1| - |z_2|$

$$|z_1 - z_2|^2 = (z_1 - z_2)(\overline{z}_1 - \overline{z}_2) = |z_1|^2 - 2\,\mathrm{RE}(z_1\cdot\overline{z}_2) + |z_2|^2$$
$$\ge |z_1|^2 - 2|z_1||z_2| + |z_2|^2 = (|z_1| - |z_2|)^2$$

$\Rightarrow |z_1 - z_2| \ge \big||z_1| - |z_2|\big|$

Note:

THE TRIANGLE INEQUALITY CAN BE EXTENDED TO ANY FINITE SUM AS FOLLOWS:

$$|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|$$

Example 5 FIND $|z|$ WHEN $z = \frac{(1+i)^4}{(1+6i)(2-7i)}$

Solution

$$|z| = \frac{|1+i|^4}{|1+6i||2-7i|} = \frac{\sqrt{1^2+1^2}^{\,4}}{\sqrt{1^2+6^2}\sqrt{2^2+(-7)^2}}$$

$$= \frac{\left(\sqrt{2}\right)^4}{\sqrt{37}\sqrt{53}} = \frac{4}{\sqrt{37}\sqrt{53}}$$

Exercise 7.4

1. PERFORM EACH OF THE FOLLOWING OPERATIONS AND WRITE YOUR ANSWER IN THE FORM OF $a+bi$ WHERE a AND b ARE REAL NUMBERS.

   A $\frac{1}{2+3i}$ B $\frac{5+4i}{2+3i}$ C $\frac{2+3i}{10-4i}$ D $\frac{2+i}{3-4i}$

   E $\overline{\left(\frac{2-3i}{4+5i}\right)}$ F $\frac{\overline{1+3i}}{\overline{4-i}}$ G $\frac{(2-3i)(4-i)}{(i-1)(i+1)}$ H $\frac{(7+i)(3-i)}{2+i}$

2. GIVEN TWO COMPLEX NUMBERS $z_1 = 3+4i$ AND $z_2 = -6\ i$, FIND EACH OF THE FOLLOWING:

   A $|z_1|$ B $|z_2|$ C $|z_1||z_2|$ D $|z_1 z_2|$

   E COMPARE THE VALUES IN **C** AND **D**.

   F $|z_1 + z_2|$, $|z_1| + |z_2|$ AND COMPARE THE TWO VALUES.

   G $|z_1 - z_2|$, $|z_1| - |z_2|$ AND COMPARE THE TWO VALUES.

   H $|z_1| - |z_2|$, $\left||z_1| - |z_2|\right|$ AND COMPARE THE TWO VALUES.

3. CAN YOU CONCLUDE THAT THE RESULT IN **E** IS TRUE FOR ANY TWO COMPLEX NUMBERS $z_1 = x + yi$ AND $z_2 = a + bi$ FOR REAL NUMBERS x, y, a, AND b?

7.4 SIMPLIFICATION OF COMPLEX NUMBERS

WITH THE HELP OF THE CONCEPTS DISCUSSED SO FAR, YOU CAN SIMPLIFY A GIVEN CO
EXPRESSION. ACTUALLY SIMPLIFICATION MEANS APPLYING THE PROPERTIES OF THE FOUR O
ON A GIVEN EXPRESSION OF COMPLEX NUMBERS AND WRITE IT IN THE FORM OF $a+bi$.

Example 1 EXPRESS THE FOLLOWING IN THE FORM OF $a+bi$.

A

$$\frac{(4+2i)(5-6i)}{(1+i)(1-3i)} = \frac{20+-24i+10i+12}{1-3i+i+3} = \frac{32-14i}{4-2i}$$

$$= \left(\frac{32-14i}{4-2i}\right)\left(\frac{4+2i}{4+2i}\right) \text{ (WHY?)}$$

$$= \frac{128+64i-56i+28}{16+4} = \frac{156+8i}{20}$$

$$= \frac{39}{5} + \frac{2}{5}i$$

B $\left(1+\sqrt{-81}\right)-(2-\sqrt{-16})+\left(3+\sqrt{196}\right)$

$$= (1+\sqrt{-1}\sqrt{81})-(2-\sqrt{-1}\sqrt{16})+(3+\sqrt{196})$$

$$= (1+9i)-(2-4i)+(3+14) = (1-2+17)+(9i+4i)$$

$$= 16+13i$$

Example 2 SOLVE $(2-3i)(x+yi) = 3$.

Solution MULTIPLYING BOTH SIDES OF THE EQUATION $(2-3i)(x+yi) = 3$ BY $(2+3i)$ (THE COMPLEX CONJUGATE) GIVES;

$$(2+3i)(2-3i)(x+yi) = 3(2+3i) \Rightarrow 13(x+yi) = 6+9i$$

$$\Rightarrow x+yi = \frac{6}{13}+\frac{9}{13}i \Rightarrow x = \frac{6}{13} \text{ AND } y = \frac{9}{13}$$

Example 3 SOLVE $(x+1)^2 = -4$.

Solution $(x+1)^2 = -4$

$$\Rightarrow (x+1) = \pm\sqrt{-4} \Rightarrow (x+1) = \pm\sqrt{(-1)\times 4}$$

$$\Rightarrow x+1 = \pm 2i \Rightarrow x = -1\pm 2i$$

$$\Rightarrow S.S = \{-1-2i, -1+2i\}$$

AN IMPORTANT PROPERTY OF COMPLEX NUMBERS IS THAT EVERY COMPLEX NUMBER HAS A SQU

Theorem 7.3

IF w IS A NON-ZERO COMPLEX NUMBER, THEN THE EQUATION $z^2 = w$ HAS A SOLUTION $z \in \mathbb{C}$.

Proof: LET $w = a+bi,\ a, b \in \mathbb{R}$. YOU WILL CONSIDER THE FOLLOWING TWO CASES.

Case 1 SUPPOSE $b = 0$. THEN IF $a > 0$, $z = \sqrt{a}$ IS A SOLUTION, WHILE, IF $a < 0$, $z = i\sqrt{-a}$ IS A SOLUTION.

Case 2 SUPPOSE $b \neq 0$. LET $z = x+yi,\ y \in \mathbb{R}$. THEN THE EQUATION $z^2 = w$ BECOMES

$$(x+yi)^2 = x^2 - y^2 + 2xyi = a+bi,$$

SO EQUATING REAL AND IMAGINARY PARTS GIVES

$$x^2 - y^2 = a \text{ AND } 2xy = b$$

HENCE $x \neq 0$ AND $y = \dfrac{b}{2x}$

THUS, $x^2 - \left(\frac{b}{2x}\right)^2 = a$

SO $4x^4 - 4ax^2 - b^2 = 0$ AND $4(x^2)^2 - 4a(x^2) - b^2 = 0$

$$\Rightarrow x^2 = \frac{4a \pm \sqrt{16a^2 + 16b^2}}{8} = \frac{a \pm \sqrt{a^2 + b^2}}{2}$$

SINCE $x^2 > 0$ YOU MUST TAKE THE POSITIVE SIGN, AS $a - \sqrt{a^2 + b^2} < 0$. HENCE

$$x^2 = \frac{a + \sqrt{a^2 + b^2}}{2} \Rightarrow x = \pm\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}$$

THEN y IS DETERMINED BY $y = \frac{b}{2x}$.

Example 4 SOLVE THE EQUATION $z^2 = 1 + i$.

Solution PUT $z = x + yi$ THEN THE EQUATION BECOMES

$$(x + yi)^2 = x^2 - y^2 + 2xyi = 1 + i$$

$$\Rightarrow x^2 - y^2 = 1 \text{ AND } 2xy = 1$$

HENCE, $x \neq 0$ AND $y = \frac{1}{2x}$. CONSEQUENTLY

$$x^2 - \left(\frac{1}{2x}\right)^2 = 1$$

$$\Rightarrow 4x^4 - 4x^2 - 1 = 0$$

$$\Rightarrow x^2 = \frac{4 \pm \sqrt{16 + 16}}{8} = \frac{1 \pm \sqrt{2}}{2}$$

$$\Rightarrow x = \pm\sqrt{\frac{1 + \sqrt{2}}{2}}$$

THEN, $y = \frac{1}{2x} = \pm\frac{1}{\sqrt{2}\sqrt{1 + \sqrt{2}}}$

HENCE, THE SOLUTIONS ARE

$$z = \pm\left(\sqrt{\frac{1 + \sqrt{2}}{2}} + \frac{i}{\sqrt{2}\sqrt{1 + \sqrt{2}}}\right)$$

Example 5 FIND THE CUBE ROOTS OF 1.

Solution YOU HAVE TO SOLVE THE EQUATION $z^3 = 1$, OR $z^3 - 1 = 0$

NOW, $z^3 - 1 = (z - 1)(z^2 + z + 1)$.

SO $z^3 - 1 = 0$ IMPLIES $z - 1 = 0$ OR $z^2 + z + 1 = 0$

BUT, $z^2 + z + 1 = 0 \Rightarrow z = \dfrac{-1 \pm \sqrt{1^2 - 4}}{2} = \dfrac{-1 \pm \sqrt{3}i}{2}$

THUS, THERE ARE 3 CUBE ROOTS OF 1, NAMELY 1, $\dfrac{-1+\sqrt{3}i}{2}$ AND $\dfrac{-1-\sqrt{3}i}{2}$.

Exercise 7.5

1 WRITE EACH OF THE FOLLOWING IN THE FORM $a + bi$ WHERE a AND b ARE REAL NUMBERS.

A $\dfrac{13}{3-2i} - \dfrac{i^3}{1+i}$ **B** $\dfrac{5}{(i-1)(2-i)(3-i)}$ **C** $i^{120} - 4i^{94} + 3i^{31}$

D $\left(2+\sqrt{-25}-\left(3-\sqrt{-216}\right)+\left(1+\sqrt{-9}\right)\right)$ **E** $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i}$

F $i^{29} + i^{42} + i$ **G** $i^{400} + 3i^{200} + 5i - 3$ **H** $\dfrac{\sqrt{-144}}{\sqrt{-121}}$

I $\left(\sqrt{-12}\right)^3$ **J** $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$

2 GIVEN $z_1 = 2 + i$, $z_2 = 3 - 2i$ AND $z_3 = \dfrac{-1}{2} + \dfrac{\sqrt{3}}{2}i$, SIMPLIFY EACH OF THE FOLLOWING:

A $z_1^{\,3} - 3z_2^{\,2} + 4z_3$ **B** $\overline{z_3}^{\,4}$ **C** $\left|3\bar{z}_1 - 4\bar{z}_2 + z_3\right|$

D $\dfrac{z_1 z_2}{z_3}$ **E** $\dfrac{z_1 z_3}{z_2}$

3 SOLVE EACH OF THE FOLLOWING EQUATIONS:

A $z^2 + 4 = 0$ **B** $z^2 + 12 = 0$ **C** $z^2 + z + 1 = 0$

D $3z^2 - 2z + 1 = 0$ **E** $z^3 = -1$ **F** $z^4 = 1$

4 PERFORM EACH OF THE FOLLOWING OPERATIONS AND COMPARE THE VALUES OBTAINED:

A $\sqrt{(-4)(-9)}$ **B** $\sqrt{-4}\sqrt{-9}$ **C** $\sqrt{(-4)(9)}$ **D** $\sqrt{-4}\sqrt{9}$

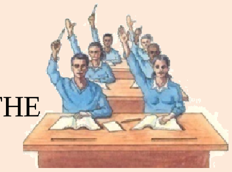
5 IF a AND b ARE ANY REAL NUMBERS: FIND CONDITIONS FOR WHICH,

$\sqrt{ab} = \sqrt{a}\sqrt{b}$ AND $\sqrt{ab} \neq \sqrt{a}\sqrt{b}$

7.5 ARGAND DIAGRAM AND POLAR REPRESENTATION OF COMPLEX NUMBERS

THIS SUB-UNIT BEGINS BY CONSIDERING THE CARTESIAN COORDINATE AXES. PREVIOUSLY, YO USED A PAIR OF NUMBERS TO REPRESENT A POINT IN A PLANE. THE MAIN TASK OF THIS SECTI SET UP A ONE-TO-ONE CORRESPONDENCE BETWEEN THE SET OF POINTS IN A PLANE AND TH COMPLEX NUMBERS. TO THIS EFFECT LET US USE THE FOLLOWING ACTIVITY AND GROUP WORK AS A STARTING POINT.

ACTIVITY 7.5



1 CONSIDER THE SET {(x, y): x AND y ARE REAL NUMBERS} IN THE COORDINATE PLANE.

- **A** LOCATE THE TWO POINTS (2, 3) AND (3, 2) IN THE CARTESIAN COORDINATE SYSTEM. DO THEY REPRESENT THE SAME POINT OR DIFFERENT POINTS? EXPLAIN.
- **B** WHEN IS (a, b) = (c, d)?
- **C** WHAT IS THE SUM (2, 3) + (5, 2)?
- **D** CAN YOU GENERALIZE THE SUM FOR (a, b) + (c, d)?

2 IDENTIFY WHETHER EACH OF THE FOLLOWING POINTS LIE ON THE x-AXIS OR THE y-AXIS.

- **A** (2, 0) **B** $(\frac{1}{2}, 0)$ **C** (0, − 3)
- **D** (0.234, 0) **E** $(x, 0);\ x \in \mathbb{R}$ **F** $(0, y);\ y \in \mathbb{R}$

NOW YOU ARE IN A POSITION TO SET UP A ONE-TO-ONE CORRESPONDENCE BETWEEN THE COMPLEX NUMBERS AND THE SET OF POINTS IN A PLANE, USING THE CORRESPONDENCE $x + yi \leftrightarrow (x, y)$.

NOTATION:

THE SET OF POINTS IN THE PLANE DENOTED BY $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ REPRESENT THE SET OF ALL ORDERED PAIRS (x, y) OF REAL NUMBERS x AND y.

Group Work 7.4

DEFINE A FUNCTION $f: \mathbb{R}^2 \to \mathbb{C}$ BY $f(x, y) = x + iy$ AND ANSWER THE FOLLOWING:

1 IF TWO POINTS (x, y) AND (a, b) WITH $(x, y) \neq (a, b)$ ARE GIVEN, THEN IS IT POSSIBLE TO HAVE $f(x, y) = f(a, b)$? EXPLAIN.

2 IF A COMPLEX NUMBER $x + yi$ IS GIVEN, THEN DOES A POINT (a, b) ALWAYS EXIST SO THAT $x + yi = f(a, b)$? EXPLAIN.

Geometric representation of complex numbers

THE COMPLEX NUMBER $z = x + yi$ IS UNIQUELY DETERMINED BY THE ORDERED PAIR OF REAL NUMBERS (x, y). THE SAME IS TRUE FOR THE POINT $P(x, y)$ IN THE PLANE WITH CARTESIAN COORDINATES x AND y. HENCE IT IS POSSIBLE TO ESTABLISH A ONE-TO-ONE CORRESPONDENC BETWEEN THE SET OF COMPLEX NUMBERS AND ALL POINTS IN THE PLANE. YOU MERELY AS THE COMPLEX NUMBER $z = x + iy$ WITH THE POINT $P(x, y)$. THE PLANE WHOSE POINTS REPRESENT THE COMPLEX NUMBERS IS CALLED THE **complex plane** OR THE **z-plane**. REAL NUMBERS OR POINTS CORRESPONDING TO $x = (x, 0)$ ARE REPRESENTED BY POINTS ON THE x-AXIS; HENCE THE x-AXIS IS CALLED THE **Real axis**. PURELY IMAGINARY NUMBERS OR POINTS CORRESPONDING TO $iy = (0, y)$ ARE REPRESENTED BY POINTS ON THE y-AXIS, AND HENCE WE CALL THE y-AXIS THE **Imaginary axis.** THE COMPLEX NUMBERS WITH POSITIVE IMAGINARY PART LIE IN THE UPPEF HALF PLANE, WHILE THOSE WITH NEGATIVE IMAGINARY PART LIE IN THE LOWER HALF PLANI

INSTEAD OF CONSIDERING THE POINT $P(x, y)$ AS THE REPRESENTATION OF $z = x + yi$, YOU MAY EQUALLY CONSIDER THE DIRECTED SEGMENT OR THE VECTOR EXTENDING FROM THE ORIGIN C P AS THE REPRESENTATION OF A COMPLEX NUMBER $z = x + yi$. IN THIS CASE, ANY PARALLEL SEGMENT OF THE SAME LENGTH AND DIRECTION IS TAKEN AS REPRESENTING THE SAME COMPLEX NUMB

FOR EXAMPLE, $z = x + yi$, $z_1 = -4 + 2i$ AND $z_2 = 2 - 3i$ CAN BE REPRESENTED AS SHOWN IN FIGURE 7.1 BELOW.

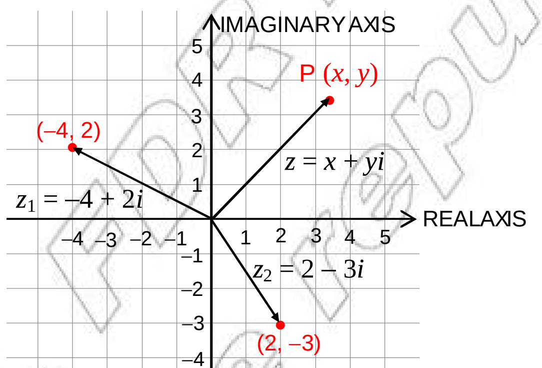


Figure 7.1

$\bar{z}$, $|z|$, AND THE SUM AND DIFFERENCE OF COMPLEX NUMBERS CAN BE PRESENTED AS FOLLOV

✓ $|z|$ IS THE LENGTH OF THE VECTOR REPRESENTING THE COMPLEX NUMBER z OR THE DISTANCE FROM THE ORIGIN TO THE POINT CORRESPONDING TO z IN THE COMPLEX PLANE. MORE GENERALLY, $|z_1 - z_2|$ IS THE DISTANCE BETWEEN THE POINTS CORRESPONDING TO z_1 AND z_2 IN THE COMPLEX PLANE.

$$|z_1 - z_2| = |(x_1 + y_1 i) - (x_2 + y_2 i)|$$

$$= |(x_1 - x_2) + (y_1 - y_2)i| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

✓ THE POINT CORRESPONDING TO $\bar{z}$ IS THE REFLECTION OF THE POINT CORRESPONDING TO z WITH RESPECT TO THE REAL AXIS.

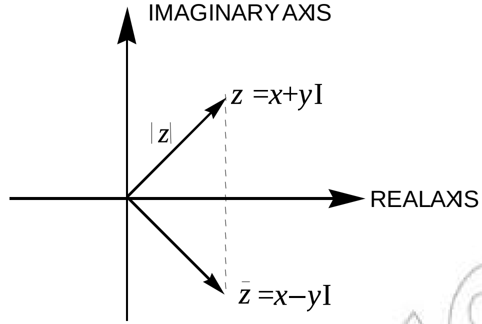


Figure 7.2

FIGURE 7.2 SHOWS THAT WHEN THE POINTS CORRESPONDING TO z AND $\bar{z}$ ARE PLOTTED ON THE COMPLEX NUMBER PLANE, ONE IS THE REFLECTION OF THE OTHER.

✓ BECAUSE OF THE EQUATION

$$(x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2)i,$$

COMPLEX NUMBERS CAN BE ADDED AS VECTORS USING THE PARALLELOGRAM LAW. SIMILA COMPLEX NUMBER $z_1 - z_2$ CAN BE REPRESENTED BY THE VECTOR FROM (x_2, y_2) TO (x_1, y_1), WHERE $z_1 = x_1 + y_1$I AND $z_2 = x_2 + y_2$I. (SEE FIGURE 7.3)

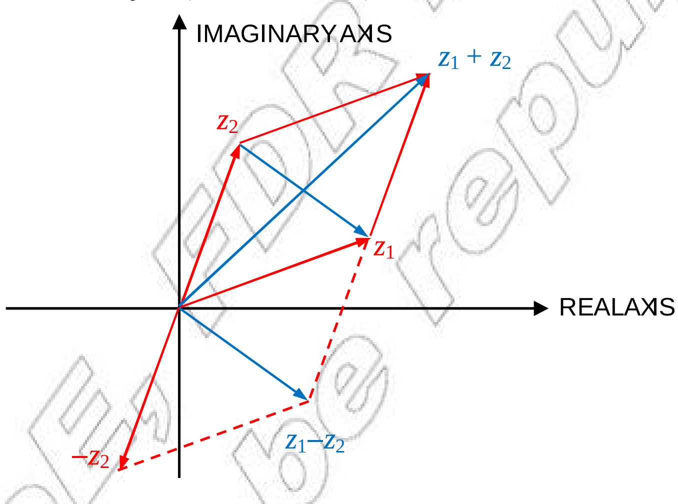


Figure 7.3 *Complex number addition and subtraction*

Polar representation of a complex number

YOU HAVE SEEN THAT A COMPLEX NUMBER CAN BE REPRESENTED AS A POINT IN THE PLANE. YOU CAN USE POLAR COORDINATES RATHER THAN CARTESIAN COORDINATES, GIVING THE CORRESPONDENCES (ASSUMING $z \neq 0$)

$$z = x + yi \leftrightarrow (x, y) \leftrightarrow (r, \ )$$

LET $z = x + yi$ BE A NON – ZERO COMPLEX NUMBER, $r = |z| = \sqrt{x^2 + y^2}$. THEN YOU HAVE $x = r\cos\theta$, $y = r\sin\theta$, WHERE θ IS THE ANGLE MADE BY THE VECTOR CORRESPONDING TO z WITH THE POSITIVE x-AXIS. θ is UNIQUE UPTO ADDITION OF A MULTIPLE OF 2π RADIANS.

FROM THE ABOVE DISCUSSIONS, YOU HAVE:

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

THIS IS CALLED THE **polar representation of z**.

Definition 7.7

WHEN A COMPLEX NUMBER IS WRITTEN IN THE FORM $z = r(\cos\theta + i\sin\theta)$, θ IS CALLED AN **argument of z** AND IS DENOTED BY **arg z**. THE PARTICULAR ARGUMENT OF z LYING IN THE RANGE $-\pi < \theta \le \pi$ IS CALLED THE **principal argument** OF z AND IS DENOTED BY **Arg z**.

FROM FIGURE 7.4, THE PRINCIPAL ARGUMENT OF z:

ARG $z = \theta$, YOU ALSO HAVE $\arg z = \theta + 2\pi$

IN GENERAL,

$$r(\cos\theta + i\sin\theta) = r(\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi))$$

FOR ANY INTEGER n, $\theta + 2n\pi$ IS ALSO AN ARGUMENT OF z, WHENEVER $\theta = \arg(z)$.

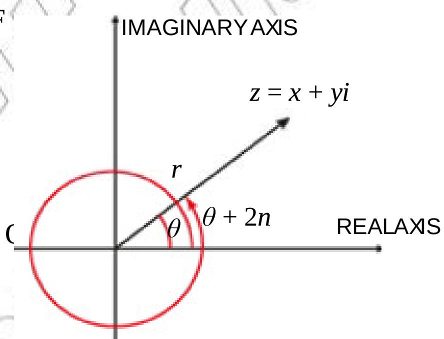


Figure 7.4

Example 1

$$\mathrm{Arg}(1) = 0,\ \mathrm{Arg}(-1) = \pi,\ \mathrm{Arg}(i) = \frac{\pi}{2},\ \mathrm{Arg}(-i) = -\frac{\pi}{2}.$$

NOTE THAT $\frac{y}{x} = \tan\theta$ IF $x \ne 0$, SO θ IS DETERMINED BY THIS EQUATION UP TO A MULTIPLE OF π.

IN FACT, $\mathrm{Arg}\,z = \tan^{-1}\left(\frac{y}{x}\right) + k\pi$,

WHERE $\begin{cases} k = 0, \text{ if } x > 0 \\ k = 1, \text{ if } x < 0, y \ge 0 \\ k = -1, \text{ if } x < 0, y < 0 \end{cases}$

Example 2

EXPRESS EACH OF THE FOLLOWING COMPLEX NUMBERS IN POLAR FORM.

A $z = 2 + 2\sqrt{3}i$ **B** $z = -5 + 5i$ **C** $z = 3i$ **D** $z = -1$

Solution

A $r = \sqrt{2^2 + \left(2\sqrt{3}\right)^2} = \sqrt{4+12} = \sqrt{16} = 4$

$= \text{TAN}^{-1}\left(\frac{y}{x}\right) = \text{TAN}^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \text{TAN}^{-1}\left(\sqrt{3}\right) = \frac{}{3} = \quad z \in$

THEREFORE $z = 2 + 2\sqrt{3}\, i = 4\left(\text{COS}\frac{}{3} + i\,\text{SIN}\frac{}{3}\right)$ IS THE POLAR FORM OF z.

B $r = \sqrt{(-5)^2 + 5^2} = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2}$

$= \text{TAN}^{-1}\left(\frac{y}{x}\right) = \text{TAN}^{-1}\left(\frac{5}{-5}\right) = \text{TAN}^{-1}(-1) = \frac{3}{4} = \quad z \in$

THEREFORE $z = 5\sqrt{2}\,(\text{COS}\frac{3}{4} + i\,\text{SIN}\frac{3}{4})$ IS THE POLAR FORM OF z.

C $r = \sqrt{0^2 + 3^2} = \sqrt{9} = 3,\ x = 0 \Rightarrow \text{COS} =$

$= \text{COS}^{-1}(0) = \frac{4n+1}{2},\ n \in Z$. IN PARTICULAR IF $n = 0$, THEN $= \frac{}{2}$.

THE PRINCIPAL ARGUMENT IS ARG$(z) = \frac{}{2}$

THEREFORE, $z = 3(\text{COS}\frac{}{2} + i\,\text{SIN}\frac{}{2})$ IS THE POLAR FORM OF z.

D $r = \sqrt{(-1)^2 + 0^2} = 1,\quad = \text{SIN}^{-1}(0)$ AND $= \text{COS}^{-1}(-1) = (2n+1),\ n \in Z$

THE PRINCIPAL ARGUMENT IS ARG$(z) =$

THEREFORE, $z = \text{COS} + i\,\text{SIN}$ IS THE POLAR FORM OF z.

> **Note:**
>
> IF $z_1 = r_1(\text{COS}_1 + i\ \text{SIN}_1)$ AND $z_2 = r_2(\text{COS}_2 + i\ \text{SIN}_2)$, THEN
>
> $z_1 = z_2 \Leftrightarrow r_1 = r_2$ AND $_1 = {}_2 + 2k,\ k \in \mathbb{Z}$. *(Why?)*

Example 3

A $3\left(\text{COS}\frac{}{3} + i\ \text{SIN}\frac{}{3}\right) = \left(3\ \text{COS}\frac{7}{3} + i\ \text{SIN}\frac{7}{3}\right) = 3\left(\text{COS}\frac{5}{3} - i\ \text{SIN}\frac{5}{3}\right)$

B $8\left(\text{COS}\frac{}{6} + i\ \text{SIN}\frac{}{6}\right) = \left(8\ \text{COS}\frac{13}{6} + i\ \text{SIN}\frac{13}{6}\right) = 8\left(\text{COS}\frac{11}{6} - i\ \text{SIN}\frac{11}{6}\right)$

THE POLAR REPRESENTATION OF A COMPLEX NUMBER IS IMPORTANT BECAUSE IT GIVES SIMPLE METHOD OF MULTIPLYING COMPLEX NUMBERS.

Theorem 7.4

SUPPOSE $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ AND $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. THEN THE FOLLOWING HOLD TRUE.

A $z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

B $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$, PROVIDED THAT $r_2 \neq 0$.

Proof:

A $z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)]$

$= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

HENCE, A IS PROVED.

THE PROOF OF B IS LEFT AS AN EXERCISE TO YOU.

FROM THE ABOVE THEOREM, IF $\theta_1 = \theta_2 = \theta$ AND $r_1 = r_2 = r$ AND WE HAVE A COMPLEX NUMBER $z = r(\cos\theta + i\sin\theta)$, THEN ONE CAN SHOW THAT:

$$z^2 = r^2(\cos 2\theta + i\sin 2\theta); \qquad \frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta)$$

SO ONE CAN GENERALIZE AS FOLLOWS:

$z^n = r^n(\cos n\theta + i\sin n\theta)$; FOR ANY INTEGER n.

Interested students may try the proof for fun!

Remark:

1 IF θ IS AN ARGUMENT OF z, THEN $n\theta$ IS AN ARGUMENT OF z^n.

2 IF θ IS AN ARGUMENT OF THE NON-ZERO COMPLEX NUMBER z, THEN $-\theta$ IS AN ARGUMENT OF z^{-1}.

3 IF θ_1 AND θ_2 ARE ARGUMENTS OF z_1 AND z_2 THEN $\theta_1 - \theta_2$ IS AN ARGUMENT OF $\dfrac{z_1}{z_2}$.

4 IN TERMS OF PRINCIPAL ARGUMENT, YOU HAVE THE FOLLOWING EQUATIONS:

I $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 + k_1 2\pi$

II $\operatorname{Arg}(z^{-1}) = -\operatorname{Arg} z + k_2 2\pi$

III $\operatorname{Arg}\left(\dfrac{z_1}{z_2}\right) = \operatorname{Arg} z_1 - \operatorname{Arg} z_2 + k_3 2\pi$

IV $\operatorname{Arg}(z_1 \ldots z_n) = \operatorname{Arg} z_1 + \ldots + \operatorname{Arg} z_n + k_4 2\pi$

V $\operatorname{Arg}(z^n) = n \operatorname{Arg}(z) + k_5 2\pi$ WHERE k_1, k_2, k_3, k_4, k_5 ARE INTEGERS.

5 IT IS NOT ALWAYS TRUE THAT $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$

FOR EXAMPLE, $\operatorname{Arg}(-1) = \pi$ *but* $\operatorname{Arg}((-1)(-1)) = \operatorname{Arg} 1 = 0 \neq \pi + \pi$

Example 4 FIND THE MODULUS AND PRINCIPAL ARGUMENT OF $z = \left(\frac{\sqrt{3}+i}{1+i}\right)^{17}$ AND HENCE EXPRESS z IN POLAR FORM.

Solution

$$|z| = \frac{\left|\sqrt{3}+i\right|^{17}}{|1+i|^{17}} = \frac{2^{17}}{\left(\sqrt{2}\right)^{17}} = 2^{\frac{17}{2}}$$

$$\text{ARG } z = 17\,\text{ARG}\left(\frac{\sqrt{3}+i}{1+i}\right) = 17\left(\text{ARG}\left(\sqrt{3}+i\right) - \text{ARG}(1+i)\right)$$

$$= 17\left(\frac{\pi}{6} - \frac{\pi}{4}\right) = \frac{-17\pi}{12}.$$

HENCE $\text{ARG } z = \left(\frac{-17\pi}{12}\right) + 2k\pi$, WHERE k IS AN INTEGER. WE SEE $k = 1$ AND HENCE

$$\text{ARG } z = \frac{7\pi}{12}$$

CONSEQUENTLY, $z = 2^{\frac{17}{2}}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$

Exercise 7.6

1. GIVE THE CORRESPONDING REPRESENTATION OF THE FOLLOWING COMPLEX NUMBERS IN THE ARGAND PLANE AS POINTS AND IDENTIFY THE QUADRANTS TO WHICH THEY BELONG:

   A $1+i$ **B** $2-3i$ **C** $3+4i$ **D** $-1-2i$

2. EXPRESS EACH OF THE FOLLOWING COMPLEX NUMBERS IN POLAR FORM; AND IDENTIFY THE QUADRANT TO WHICH IT BELONGS; FIND THE MODULUS FOR EACH:

   A 3 **B** $3i$ **C** -3

   D $-3i$ **E** $2+2\sqrt{3}\,i$ **F** $2\sqrt{2}-2\sqrt{2}\,i$

   G $-\sqrt{6}-\sqrt{2}\,i$ **H** $\frac{\sqrt{3}}{2}-\frac{3}{2}i$

3. GIVE THE CORRESPONDING COMPLEX NUMBER FOR EACH OF THE FOLLOWING POLAR REPRESENTATIONS:

   A $(5, \frac{\pi}{3})$ **B** $(6, \frac{\pi}{6})$ **C** $(5, \frac{\pi}{4})$ **D** $(5, 4\frac{\pi}{3})$

4. FIND THE PRINCIPAL ARGUMENT FOR EACH OF THE FOLLOWING:

   A $z = 4 + 3i$ **B** $z = 4 - 3i$ **C** $z = -2 + 2i$ **D** $z = -2 - 2i$

Key Terms

Argand diagram	complex plane	modulus
argument	conjugate	polar form
complex number	imaginary axis	real axis

Summary

1 AN EXPRESSION OF THE FORM $x + yi$ IS CALLED A **complex number**, WHERE x AND y ARE REAL NUMBERS, AND $i^2 = -1$. IN THIS EXPRESSION THE NUMBER x IS CALLED THE **real part** OF z; AND y IS CALLED THE **imaginary part** OF z.

2 A COMPLEX NUMBER $x - yi$ IS CALLED THE **conjugate** OF A COMPLEX NUMBER $x + yi$.

3 IF $z = x + yi$, THEN ITS CONJUGATE DENOTED BY $\bar{z}$ IS GIVEN BY $\bar{z} = x - yi$; THE **modulus of z** DENOTED $|z|$ IS GIVEN BY $|z| = \sqrt{x^2 + y^2}$.

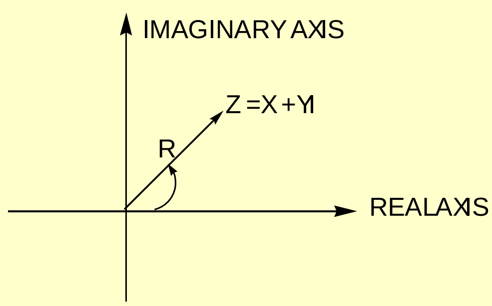


Figure 7.5

4 LET $(r, \ )$ BE THE POLAR COORDINATES OF THE POINT REPRESENTING THE COMPLEX NUMBER $z = x + yi$, $r \geq 0$. THEN,

$$x = r\text{COS}\ , \quad y = r\text{SIN}\ , \quad r = |z| = \sqrt{x^2 + y^2} \quad \text{AND} \quad = \text{TAN}^{-1}\left(\frac{y}{x}\right), \text{ for } x \neq 0$$

$z = r\text{COS} + ir\ \text{SIN} = r(\text{COS} + i\ \text{SIN}\ )$ IS CALLED THE **polar representation** OF z.

5 THE ANGLE IS CALLED THE **argument** OF Z AND WE WRITE IT AS ARG(z).

6 SINCE $r(\text{COS} + i\ \text{SIN}\ ) = r(\text{COS}(+ 2n\ ) + i\ \text{SIN}(+ 2n\ ))$ FOR ANY INTEGER n, THEN $+ 2n$ IS ALSO AN ARGUMENT OF z FOR ANY INTEGER n WHENEVER = ARG(z).

7 ARG(Z) IS CALLED THE **principal argument** OF Z; IT IS THE VALUE OF THE ARGUMENT OF Z IN THE INTERVAL $(-\pi, \pi]$, THAT IS, $-\pi < \text{Arg}(z) \le \pi$.

8 IF $\text{Arg}(z)$ IS THE PRINCIPAL ARGUMENT OF z, THEN $\arg(z) = \text{Arg}(z) + 2n\pi$, $n \in \mathbb{Z}$ DESCRIBES ALL POSSIBLE VALUES OF $\arg(z)$.

Review Exercises on Unit 7

1 IN EACH OF THE FOLLOWING, SOLVE FOR x AND y.

A $x + yi = i(4 - 3i)$

B $\dfrac{1+2i}{x+yi} = 1 - \sqrt{-4}$

C $(3+i)(x + yi)(3 + 4i) = 3 + 9i$

D $(2x + yi)(i + 4) = \dfrac{1}{3+5i}$

E $2x + 3xi + 2y = 28 + 9i$

2 GIVEN THE COMPLEX NUMBER $z = 3 - 4i$:

A FIND THE CONJUGATE OF z.

B FIND THE MODULUS OF z.

C FIND THE MODULUS OF THE CONJUGATE.

D EXPRESS z IN POLAR FORM.

3 FIND THE CONJUGATE, ARGUMENT AND MODULUS OF EACH OF THE FOLLOWING EXPRESS

A $\dfrac{3+i}{5-4i}$

B $\dfrac{(2-3i)(4+i)}{\left(i\sqrt{3}+1\right)\left(\frac{1}{2}i+5\right)}$

C $\dfrac{(i+2)(3-4i)(5+3i)}{(2i+1)(4i+3)(5i-3)}$

4 SIMPLIFY EACH OF THE FOLLOWING AND WRITE EACH IN THE FORM $a + bi$ WHERE a AND b ARE REAL NUMBERS.

A $i^{320} - 5i^{121} + 3i^{45}$

B $\dfrac{1+2i}{6-8i} + \dfrac{6-2i}{10i}$

C $i + (i + 1)^2 + (i - 1)^3 + (i + 2)^4$

D $\dfrac{4x}{1-6xi} - \dfrac{2i}{3-i}$

E $\left(\dfrac{1-i}{\sqrt{2}}\right)^{40}$

F $(1-i)^{80}$

G $\left(\dfrac{i-\sqrt{3}}{1-i}\right)^{30}$

5 IF z_1, z_2 ARE COMPLEX NUMBERS, THEN PROVE THAT

$$|z_1+z_2|^2+|z_1-z_2|^2=2\left(|z_1|^2+|z_2|^2\right)$$

6 SOLVE EACH OF THE FOLLOWING EXPRESSIONS OVER $\mathbb{C}$.

A $x^3+2x^2+x-4=0$ **B** $x^2+2x+3=0$

C $x^3-2x^2-3x+10=0$ **D** $x^4+2x^2+2=0$

7 EXPRESS EACH OF THE FOLLOWING IN POLAR FORM AND EVALUATE z^{10} FOR EACH

A $z=4+4\sqrt{3}\,i$ **B** $z=3\sqrt{2}-3\sqrt{2}\,i$

C $z=-2\sqrt{6}-2\sqrt{2}\,i$ **D** $z=\frac{\sqrt{3}}{5}-\frac{3}{5}i$

E $z=1-i\sqrt{3}$ **F** $z=-\sqrt{3}+i$

8 WRITE THE MULTIPLICATIVE INVERSE FOR EACH OF THE FOLLOWING COMPLEX NUMBERS AND WRITE THE ANSWERS IN THE FORM OF $a+bi$.

A $\frac{2+3i}{1+i}$ **B** $\frac{5-7i}{2+10i}$ **C** $\frac{3+2i}{7-\sqrt{5}i}$

9 DESCRIBE EACH OF THE FOLLOWING GEOMETRICALLY IN A COMPLEX PLANE.

A $|z-1|=1$ **B** $|z-1|<1$ **C** $|z-1|>1$

10 CONVERT EACH OF THE FOLLOWING FROM POLAR TO CARTESIAN

A $\sqrt{2}\left(\text{COS}\frac{3}{4}-i\ \text{SIN}\frac{3}{4}\right)$ **B** $\sqrt{2}\left(\text{COS}\frac{3}{4}+i\ \text{SIN}\frac{3}{4}\right)$

C $\text{COS}-i\ \text{SIN}$ **D** $5\left(\text{COS}\frac{2}{3}-i\ \text{SIN}\frac{2}{3}\right)$